NOTES

ON

ELECTRICITY AND MAGNETISM

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ELECTRICITY.

1. Coulomb's Law.

If there be two point charges e_1 and e_2, separated by a distance r, the mutual force exerted between them will be $\frac{e_1 e_2}{r^2}$.

2. Coulomb's Torsion Balance.

This instrument was designed by Coulomb for verifying the truth of the above law. It consists of an insulating rod AB (Fig. 1a) suspended from a torsion head by a fine wire and having a small gilt pith-ball fixed at A. C is a similar pith-ball fixed at the end of an insulating rod. The balls are adjusted so as to touch each other and are then charged. The ball A is repelled through an angle θ_0 until the force of repulsion is balanced by the torsion in the suspending wire; the torsion is then a measure of the force of repulsion.

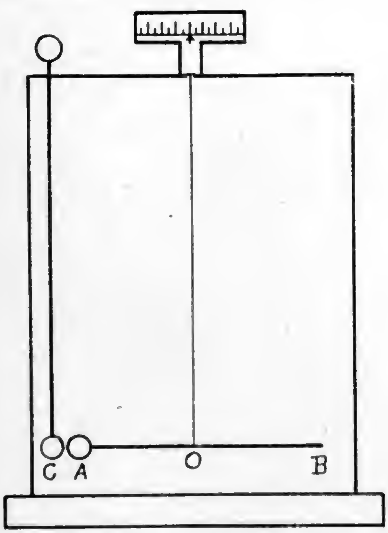

Fig. 1a

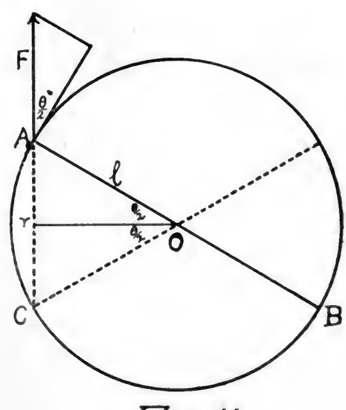

Fig. 1b

According to Coulomb's law $Fr^2 = e_1 e_2 =$ a constant; hence we can test the truth of this law by increasing the torsion and observing whether the distance between the balls varies according to the law. Let the torsion head be turned back through $T°$ thereby reducing the angle subtended by the balls at 0 to θ, and making

the torsion in the wire equal to $(T+\theta)$. Then from Fig. 1b, $CA=2l \sin\frac{\theta}{2}$. The component of F, the force of repulsion between C and A, tending to twist the wire will be $F\cos\frac{\theta}{2}$, and the moment of this component about 0 is $Fl\cos\frac{\theta}{2}$. This moment is balanced by the torsion in the wire.

If K be the moment of the couple required to produce a torsion of 1° in the wire, for a torsion of $(T+\theta)°$ the couple will be $K(T+\theta)$.

$$\therefore\ Fl\cos\frac{\theta}{2}=K(T+\theta)$$

$$\therefore\ F=\frac{K(T+\theta)}{l\cos\frac{\theta}{2}}$$

Then from Coulomb's law

$$\begin{aligned} Fr^2 &= \text{a constant} \\ &= \frac{K(T+\theta)}{l\cos\frac{\theta}{2}}\,4l^2\sin^2\frac{\theta}{2} \\ &= 4Kl(T+\theta)\sin\frac{\theta}{2}\tan\frac{\theta}{2}. \end{aligned}$$

Hence, by varying T we may test the truth of the law.

3. Unit of Electricity.

By Coulomb's law the force between two point charges of electricity e_1 and e_2, distant r cms. apart is given by

$$F=\frac{e_1 e_2}{r^2}.$$

Let the charges be equal and placed 1 cm. apart, then

$$F=e^2.$$

Therefore, to obtain unit force between the charges each must be a unit of charge. Hence, in the C.G.S. system, we have the following definition:

Unit charge of electricity is such that when placed in a vacuum at a distance of 1 cm. from an equal and similar charge it is repelled with a force of 1 dyne.

4. Applications of Coulomb's Law to Magnetic and to Gravitational Matter.

In magnetism, Coulomb's law may be expressed as

$$F=\frac{m_1 m_2}{r^2}$$

where m_1 and m_2 represent pole-strengths. This law may be verified

by means of the torsion balance if magnetic poles are substituted for the charged pith-balls.

By a similar argument to that used above we may arrive at a definition of unit magnetic pole as being that pole-strength which will exert a force of 1 dyne on an equal and similar pole distant 1 cm. from it.

Similarly, for gravitational matter, Coulomb's law gives the relation

$$F = \frac{M_1 M_2}{r^2}$$

where M_1 and M_2 represent the respective masses of two bodies. The truth of this law may be shown by an application of Coulomb's torsion balance, which is commonly known as the Cavendish experiment, in which two small silver balls are attracted by two large lead balls. For a complete description of this experiment read the chapter on gravitation in Poynting and Thomson's "Properties of Matter".

From this gravitational law we may derive the definition of unit mass as being that mass which will exert a force of 1 dyne on an equal mass at a distance of 1 cm. This is known as the "astronomical unit of mass" and must not be confused with the ordinary units such as the gram or the pound.

5. Electric Intensity at a Point.

The electric intensity at a point is the force exerted on a unit charge placed at that point.

Theorem—The electrical intensity due to an equipotential surface is everywhere normal to the surface.

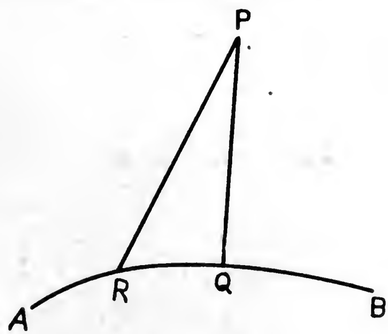

Fig 2

Let AB (Fig. 2) represent an equipotential surface; consider the electric intensity at P due to AB. Draw PQ perpendicular to AB, and let R be any point on AB.

Now the points R and Q are at the same potential, therefore, by section 14, the quantities of work required to take a unit charge

along the paths PQ and PR are equal. But the path PR is equivalent to $PQ+QR$; therefore the work done in taking the charge along QR is zero. Hence the force due to AB must be normal to the surface, for if not, it would have a component which would oppose or assist the motion of the charge along QR, thereby making the work along this path not equal to zero.

A similar argument will hold for any point of AB; therefore the electrical intensity is everywhere normal to the surface.

6. Solid Angles.

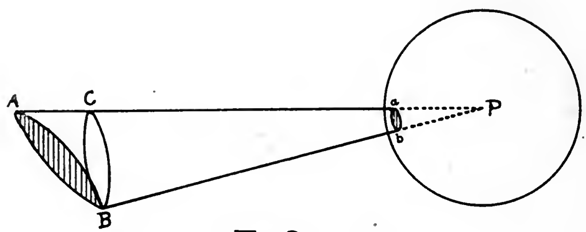

Fig. 3

Let AB (Fig. 3) represent a surface, and P any point not lying on it. From every point on the boundary of AB draw lines to P, thus forming a cone with P as apex. Then the surface AB is said to subtend a solid angle at P, the angle being bounded by the cone.

In order to measure the numerical value of this angle, with P as centre draw a sphere of unit radius, on the surface o. which the cone will intercept an area ab. This area is a measure of the solid angle. If a sphere of radius PB is described about P, the cone will intercept on it an area BC, and, as will be readily seen, the surfaces AB and BC subtend the same solid angle at P. Now the areas of the surfaces ab and BC are proportional to the squares of the radii of their generating spheres, so that, if ω is the solid angle subtended at P by the surface AB,

$$\frac{\text{Area } BC}{PB^2} = \text{Area } ab = \omega$$

Where ω is small, BC may be regarded as the orthogonal projection of AB. Then if the angle between the normals to the two surfaces is θ.

$$\text{Area } BC = \text{Area } AB \times \cos\theta$$

$$\text{Therefore } \omega = \frac{\text{Area } AB \times \cos\theta}{PB^2}.$$

7. Attraction Due to a Uniform Circular Disc at a Point on its Axis.

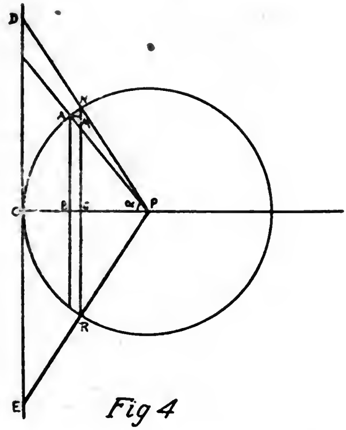

Fig 4

Let DOE (Fig. 4) represent a section through the centre of the disc, and let P be a point on the axis OP. With P as centre and PO as radius describe a sphere, and join DP and EP cutting the circular section of the sphere at N and R.

By section 6, the solid angles subtended at P by the disc DOE and the section of the sphere NOR are equal. Now the centres of the two sections are at a common distance OP from the point P, therefore it is evident that in considering the attraction at the point P we may replace the disc DOE by the section of a sphere NOR, provided that the density of charge, ρ, is the same on each.

Take now a small element AN on the circumference of the circular section and draw AB and NC perpendicular to the axis OP. If the element AN is rotated about OP, keeping the lengths AB and NC fixed, it will trace out an annular ring on the surface of the sphere The area of this elemental ring will be given by

$$\text{Area} = 2\pi AB \,.\, AN$$

$$= 2\pi AB \,.\, \frac{AM}{\sin OPA}$$

since, angle $ANM =$ angle OPA

Now, $AB = AP \sin OPA$.

$$\therefore \text{ area of ring} = 2\pi AM \,.\, AP \frac{\sin OPA}{\sin OPA}$$

$$= 2\pi AP \,.\, AM.$$

The total area of the spherical cap NOR will thus be obtained by taking successive elements of the surface from N to O, and therefore, successive values of AM from C to O.

$$i.e.,\ \text{total area} = 2\pi AP(OP-PB)$$

$$= 2\pi AP^2\left(\frac{OP}{AP} - \frac{PB}{AP}\right)$$

$$= 2\pi AP^2(1-\cos\alpha)$$

since $OP = AP$,

and $\dfrac{PB}{AP} = \cos OPA = \cos\alpha.$

$$\text{Attraction at } P \text{ due to the disc} = \frac{2\pi\rho AP^2(1-\cos\alpha)}{AP^2}$$

$$= 2\pi\rho(1-\cos\alpha).$$

If the disc be infinitely large, or if the point P is taken very close to the disc, $\alpha = 90°$, and $\cos\alpha = 0$,

$\therefore$ in this case the attraction $= 2\pi\rho$.

8. Electric Intensity at the Surface of a Charged Conductor.

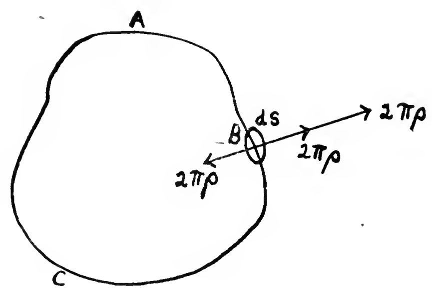

Fig. 5

Let ABC (Fig. 5) be a charged conductor of any form. At B take a circular element of area ds, so small that its surface is plane. Consider the force exerted by this little disc alone on a unit charge close to its centre. On the outside there would be a force of $2\pi\rho$ acting outwards. Similarly there would be a force of $2\pi\rho$ on the inside acting inwards. But we know that for the whole conductor there can be no force at a point inside, consequently there must be a force exerted by the rest of the conductor exactly equal and opposite to the inward force $2\pi\rho$. This is equivalent to a second outward force of $2\pi\rho$. Consequently at any point on the surface of a charged conductor,

$$R = 2\pi\rho + 2\pi\rho$$
$$= 4\pi\rho$$

9. MECHANICAL FORCE PER UNIT AREA OF A CHARGED CONDUCTOR.

From the preceding section the force per unit charge just at the surface of ds due to the charge on ds is $2\pi\rho$. Now the total charge on ds is ρds, so that the force on the area ds

$$= \rho ds \times 2\pi\rho$$
$$= 2\pi\rho^2 ds$$

or, force per unit area $= 2\pi\rho^2$.

This force is of the form of a hydrostatic pressure and tends to cause the conductor to expand.

10. THE ABSOLUTE ELECTROMETER.

The instrument usually employed for measuring potential differences is the voltmeter. The theory of this instrument depends on the magnetic action of a current. It is therefore important for theoretical reasons to have some instrument for measuring potential differences which depends only on the fundamental laws of electrostatics. This condition is satisfied by the Kelvin absolute electrometer.

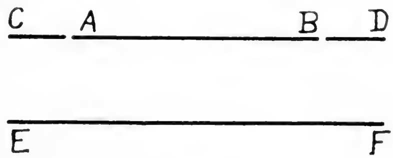

Fig. 6

In Figure 6 AB represents a brass disc suspended from one arm of a balance, and surrounded by a brass guard-ring CD. Below this is another plate EF. If AB and EF be maintained at a constant difference of potential there will be a downward pull on AB which may be measured with the balance. If m grams be the mass which it is necessary to place in the scale pan in order to restore equilibrium, the force pulling downward on AB is mg dynes. From this force the difference of potential between the plates may be calculated as follows:

Let A = Effective area of AB
$\quad$ = mean of areas of AB and of aperture in CD
d = distance between AB and EF
V = potential difference between AB and EF
ρ = density of charge on AB and EF
R = intensity of field of force
f = pull on AB in dynes

Then $f = 2\pi\rho^2 A$ by section 9. . . . . (1)

$$\text{also } R = \frac{V}{d} = 4\pi\rho \text{ by section 8,}$$

$$\text{therefore } \rho = \frac{V}{4\pi d}$$

$$\text{Substituting in (1) } f = 2\pi A \left(\frac{V}{4\pi d}\right)^2 = \frac{V^2 A}{8\pi\, d^2}$$

$$\text{Therefore } V = d\sqrt{\frac{8\pi f}{A}}$$

$$= d\sqrt{\frac{8\pi\, mg}{A}} \text{ electrostatic units of potential.}$$

11. Electric Intensity Due to a Spherical Shell at an Internal Point.

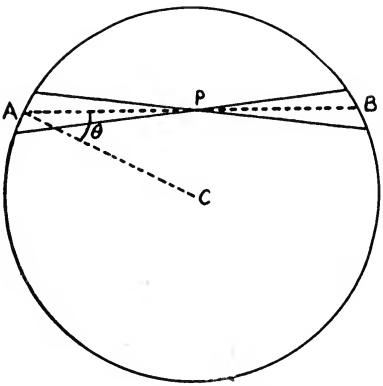

Fig 7

Let P (Fig. 7) represent an internal point in a hollow spherical shell attracting according to Coulomb's Law. Through P draw a cone of small vertical solid angle $d\omega$, so as to intercept areas ds_1 and ds_2 on the surface of the sphere at A and B respectively. Let $PA = r_1$, $PB = r_2$, where PA and PB are measured along the axis of the cone. If C is the centre of the sphere, AC represents the normal to the element ds_1. Let the angle $PAC = \theta$.

Now by the theory of solid angles

$$d\omega = \frac{ds_1 \cos\theta}{r_1^2} = \frac{ds_2 \cos\theta}{r_2^2}$$

$$\text{Therefore } ds_1 = \frac{r_1^2\, d\omega}{\cos\theta}$$

$$\text{and } ds_2 = \frac{r_2^2\, d\omega}{\cos\theta}$$

The attraction of ds_1 at P is given by $\frac{\rho ds_1}{r_1^2}$ where ρ is the surface density.

$$\text{But } \frac{\rho ds_1}{r_1^2} = \frac{\rho r_1^2 d\omega}{r_1^2 \cos\theta} = \frac{\rho d\omega}{\cos\theta}$$

Also the attraction of ds_2 at $P = \frac{\rho d\omega}{\cos\theta}$ in the opposite direction. Therefore the resultant attraction is zero.

Now the whole surface may be divided into similar pairs of elements by a series of cones through P, hence the attraction of a spherical shell at an internal point is zero.

12. Electric Intensity Due to a Spherical Shell at an External Point.

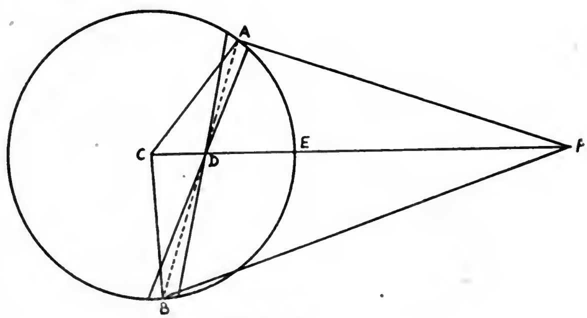

Fig. 8

Let AEB (Fig. 8) represent a spherical shell attracting according to Coulomb's Law, and P a point outside it. Let C be the centre of the shell; join CP cutting the sphere at E, and on it take an internal point D such that $\frac{CD}{CE} = \frac{CE}{CP}$.

Through D draw a cone of small vertical solid angle $d\omega$ so as to intercept areas ds_1 and ds_2 on the surface of the sphere at A and B respectively.

Let angle CAD = angle $CBD = \theta$.

$$\text{Then } d\omega = \frac{ds_1 \cos\theta}{r_1^2} = \frac{ds_2 \cos\theta}{r_2^2}$$

where r_1 and r_2 are respectively equal to DA and DB

$$\text{Therefore } ds_1 = \frac{r_1^2 d\omega}{\cos\theta}$$

$$ds_2 = \frac{r_2^2 d\omega}{\cos\theta}$$

$$\text{Attraction at } P \text{ due to } ds_1 = \frac{\rho r_1^2\, d\omega}{\cos\theta \,.\, \overline{PA}^2}$$

$$\text{Attraction at } P \text{ due to } ds_2 = \frac{\rho r_2^2\, d\omega}{\cos\theta \,.\, \overline{PB}^2}$$

Now by similar triangles, angle APC = angle $CAD = \theta$.

Similarly angle BPC = angle $CBD = \theta$.

Therefore angle APC = angle BPC.

$$\text{Also} \quad \frac{r_1}{PA} = \frac{DA}{PA} = \frac{CA}{PC}$$

$$\text{and} \quad \frac{r_2}{PB} = \frac{DB}{PB} = \frac{CB}{PC} = \frac{CA}{PC}$$

Therefore attractions of elements at A and B are equal, and therefore the resultant attraction is along PC.

$$\text{Attraction due to } ds_1 \text{ along } PC = \frac{\rho d\omega}{\cos\theta} \,.\, \frac{\overline{CA}^2}{\overline{PC}^2} \,.\, \cos\theta$$

$$= \rho\, d\omega \,.\, \frac{\overline{CA}^2}{\overline{PC}^2}$$

$$\text{Also attraction due to } ds_2 \text{ along } PC = \rho d\omega \frac{\overline{CA}^2}{\overline{PC}^2}$$

Therefore total pull on P due to two elements of surface is

$$2\rho\, d\omega \frac{\overline{CA}^2}{\overline{PC}^2}$$

$$\text{Therefore total pull on } P \text{ due to whole surface} = 2\rho \,.\, \frac{\overline{CA}^2}{\overline{PC}^2} \,.\, \Sigma d\omega$$

$$= \frac{4\pi\rho \,.\, \overline{CA}^2}{\overline{PC}^2}$$

Now $4\pi\rho\overline{CA}^2$ = total charge on sphere $= E$ and, if $PC = r$ = distance of P from the centre of the sphere,

$$\text{attraction} = \frac{E}{r^2}$$

That is, for an external point, a spherical shell of electricity acts as though its charge were concentrated at the centre.

We know that Coulomb's law holds for gravitational matter as well as for electrical, therefore if we have a thin spherical shell of matter it may be proved by the same argument as that presented above that it acts at external points as if its mass were concentrated at the centre. Now a solid sphere may be looked upon as being made up of an infinite number of concentric spherical shells, each

of which acts as if its mass were concentrated at the centre. Therefore a solid sphere of matter exerts a gravitational force at external points as though its whole mass were concentrated at a point at its centre.

13. Absolute Potential.

The absolute potential of a charged body is the work which would be done in bringing unit positive charge of electricity from an infinite distance up to the body.

14. Difference of Potential.

The difference of potential which exists between two charged bodies would, according to the above definition, be the difference between the amounts of work done in the two cases in bringing up a unit positive charge from infinity; or, more simply, the difference of potential between two charged bodies is the amount of work required to take a unit positive charge from the body of lower potential to the one of higher potential.

15. Potential of a Spherical Layer of Electricity.

We know that a charge in the form of a spherical shell acts as if the whole charge were concentrated at a point at the centre of the shell (Sec. 12), consequently in calculating the electrical potential of a sphe ical conductor bearing a charge E we assume that we are dealing with a charge E situated at the centre of the conductor.

Inside a charged spherical conductor there can be no force, hence no work is done in moving unit charge from one point to another. This tells us that all points within the conductor are at the same potential, therefore in calculating this potential we need only determine the work done in bringing a unit charge up to a point on the surface of the spherical conductor.

Fig. 9

Let O (Fig. 9) be the centre of the spherical layer of electricity, P a point at the surface and Q any point in space.

Let $OP=a$, and $OQ=x$, and let PQ be divided into n parts, n being large.

Let d be $\frac{1}{n}$ of PQ, *i.e.*, $d = \frac{x-a}{n}$.

If E is the charge on the shell, the attraction on unit charge at a point M on $PQ = \frac{E}{(a+rd)^2}$, r being some integer, such that $rd=PM$.

Therefore, at M the work done in taking unit charge over a distance d lies between $\frac{Ed}{\{a+(r-1)d\}^2}$ and $\frac{Ed}{(a+rd)^2}$

$$\text{Now}\quad \frac{d}{(a+rd)^2} < \frac{d}{\{a+(r-1)d\}(a+rd)} < \frac{d}{\{a+(r-1)d\}^2}$$

$$\text{or,}\quad \frac{d}{\{a+(r-1)d\}(a+rd)} > \frac{d}{(a+rd)^2} \text{ and } < \frac{d}{\{a+(r-1)d\}^2}$$

Then the work done in going a distance d at the point M

$$= \frac{Ed}{\{a+(r-1)d\}(a+rd)}$$

and work done in going from Q to P

$$= W = \Sigma_{r=n}^{r=1} \frac{Ed}{\{a+(r-1)d\}(a+rd)}$$

$$= \Sigma_{r=n}^{r=1} E \left\{\frac{1}{a+(r-1)d} - \frac{1}{a+rd}\right\}$$

$$= E\left[\frac{1}{a} - \frac{1}{a+d} + \frac{1}{a+d} - \frac{1}{a+2d} + \frac{1}{a+2d} - \dots - \frac{1}{a+nd}\right]$$

$$= E\left(\frac{1}{a} - \frac{1}{a+nd}\right)$$

$$= E\left(\frac{1}{a} - \frac{1}{x}\right), \text{ and, if } x \text{ is infinite,}$$

$$W = \frac{E}{a},$$

$$i.e.\ V = \frac{E}{a}.$$

Similarly, for a spherical shell of matter, the gravitational potential at the surface is given by $V = \frac{M}{a}$, *i.e.*, the mass of the shell divided by the radius.

It was proved above that a solid sphere of matter acts as though its whole mass were concentrated at the centre, hence in this case also, the gravitational potential at the surface of the sphere is given by $V = \frac{M}{a}$, where M is the whole mass of the sphere.

16. CAPACITY.

If two insulated conductors are placed in contact with each other and given a charge, they will be found to be raised to the same potential, but in general they will not have equal charges. The charge required to raise any conductor to a given potential

depends upon its dimensions, and upon its position relative to neighbouring conductors. The ratio between the charge and the acquired potential is termed the capacity of the conductor.

$$\text{That is, Capacity} = \frac{\text{Quantity}}{\text{Potential}}.$$

Therefore the capacity of a conductor is the amount of electricity required to raise its potential by one unit. Hence it follows that a conductor has unit capacity when unit charge is required to give it unit potential.

It was proved above that the potential of a charged spherical conductor is given by $V = \frac{E}{a}$, where E is the quantity of electricity on the conductor.

$$\text{But } C = \frac{E}{V}$$

$$= E \cdot \frac{a}{E}$$

$$= a.$$

Hence the capacity of a spherical conductor is numerically equal to its radius, and therefore a spherical conductor of 1 cm. radius has unit capacity.

17. Energy of a Charged Conductor.

By experiment we find that for a single isolated conductor the relation connecting charge and potential is given by

$$E = CV,$$

where C, as already shown, is a constant and is called the capacity of the conductor.

If we add to the charge a small quantity dE we raise the potential by a small amount dV such that

$$dE = CdV.$$

By definition the potential of a conductor is the measure of the work done in bringing up a unit positive charge from an infinite distance, therefore if we bring up a charge dE when the conductor is at a potential V, the work done is

$$dW = VdE$$

$$= CVdV.$$

Then the total work done in giving to the conductor a charge E would be the sum of all such expressions as V increases from 0 to V, the final value;

$$i.e., \; W = \Sigma_{V=0}^{V=V} CVdV.$$

This summation can be expressed graphically as follows:

Bring up to the conductor a charge dE, thereby raising the potential by an amount dV. The amount of work done is $dEdV$

and is represented by the area of the small rectangle $OABC$ (Fig. 10a). Bring up a second charge dE and the work done is $dE \times 2dV$,

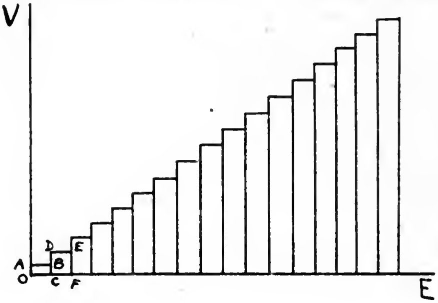

Fig. 10a

represented by the area of the rectangle $CDEF$. This process is repeated until a total charge E has been given to the conductor and the potential raised by an amount V. The total work done is then the sum of the areas of all the small rectangles, and if the elements dE and dV are taken smaller and smaller this total area

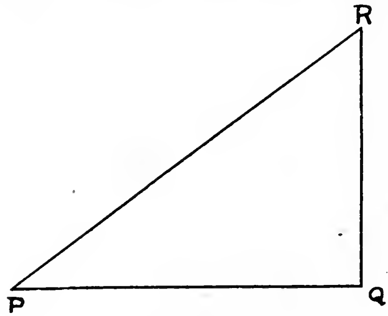

Fig. 10b

becomes a right angle triangle as shown by PQR (Fig. 10b). The height of the triangle will be $\sum_{V=0}^{V=V} dV = V$, and the length of the base will be $\sum_{E=0}^{E=E} dE = E$,

$$\therefore \text{area of } PQR = W = \tfrac{1}{2} EV,$$

and the potential energy of the conductor must be equal to the work done in charging it,

$$\therefore \text{Energy} = \tfrac{1}{2} EV.$$

18. Determination of the Elemental Charge.

When a liquid is sprayed through an atomiser so as to form minute drops it is found that most of the drops acquire a charge

of electricity, either positive or negative. Prof. Millikan undertook to measure the charges on such drops in order to find whether or not there was any uniformity in the amounts of electricity on the different drops. His method was to get the drops between two parallel metal plates (see Fig. 11), one of which could be charged

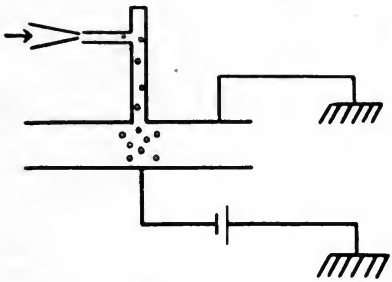

Fig. 11

either positively or negatively by means of a battery, the other being connected to earth. A microscope was focussed on a particular drop and its velocity measured by means of an attached scale, first with the plates uncharged, then with a known difference of potential between the plates. If

V = pot. diff. between the plates,

d = distance in cms. between the plates,

then the electric force $= X = \dfrac{V}{d}$.

The mathematical determination of the charge on the drop is based on Stoke's law that for a falling spherical drop of small dimensions the velocity is constant instead of increasing with time and is given by

$$6\pi\mu a V = \tfrac{4}{3}\pi a^3 g(\rho - \sigma)$$

where V = terminal velocity

μ = coefficient of viscosity

a = radius of drop

ρ = density of drop

σ = density of medium

or, if σ is negligible, and water drops are used.

$$6\pi\mu a V = \tfrac{4}{3}\pi a^3 g.$$

To get the mass of the drop,

$$a^2 = \tfrac{9}{2}\frac{\mu V}{g}.$$

Now μ was found to be 1.8×10^{-4}

$$\therefore a^2 = \tfrac{9}{2} \times \frac{1.8 \times 10^{-4}}{981} \,.\, V$$

$$\therefore\ m = \tfrac{4}{3}\,\pi a^3 = \tfrac{4}{3}\,\pi \left(\tfrac{9}{2} \times \frac{1.8 \times 10^{-4}}{981} \,.\, V \right)^{3/2}$$

$$= 3.1 \times 10^{-9}\, V^{3/2}.$$

From the equation given above, the terminal velocity is given by $V_1 = \frac{2}{9}\,\frac{ga^2}{\mu}$ under the action of gravity alone. When an electric field of value X is applied, the acceleration of the drop is changed from g to f.

Then $mf = Xe + mg$,

where e is the charge on the drop.

$$\therefore f = \frac{Xe}{m} + g.$$

Then the new terminal velocity is given by

$$V_2 = \tfrac{2}{9}\,\frac{a^2}{\mu}\left(\frac{Xe}{m} + g\right)$$

$$\therefore\ \frac{V_1}{V_2} = \frac{mg}{Xe + mg}$$

$$\therefore\ XeV_1 = mg(V_2 - V_1)$$

$$\text{or, } e = mg\left(\frac{V_2 - V_1}{XV_1}\right)$$

$$= 3.1 \times 10^{-9}\ \frac{g}{X}\,(V_2 - V_1)\,V_1^{1/2}$$

It was found that e always came out to a minimum value or some simple multiple of this value, consequently it was assumed that this was the element of electrical charge.

The values found for e by this method and others are given below.

	Value of e
Millikan	4.7×10^{-10} e.s. units
Wilson	3.1×10^{-10} "
Perrin	4.7×10^{-10} "
Rutherford	4.65×10^{-10} "

This last value is considered to be the most accurate and is the one used in all determinations involving e.

[For a full account of this work read *Science*, Sept. 30, 1910, p. 436.]

19. Electrolysis.

A conductor of electricity which is decomposed by the passage of an electric current through it is called an electrolyte. When an electrolyte is decomposed the constituent parts set free move in opposite directions under the influence of the electric field and are called ions.

The most general definition of an ion is that it is any charged body which is free to move under the influence of an electric field.

20. Faraday's Laws of Electrolysis.

1. The quantity of an electrolyte decomposed by the passage of an electric current is directly proportional to the quantity of electricity which passes through it.

2. If the same quantity of electricity passes through different electrolytes the weights of the different ions deposited will be proportional to the chemical equivalents of the ions.

The weight of an element which is set free by the passage of one coulomb of electricity through an electrolyte is called the electrochemical equivalent of the element.

21. Determination of the Mass of the Hydrogen Atom.

By experiment the electrochemical equivalent of hydrogen is .000010384.

$\therefore$ 1 coulomb of electricity sets free .000010384 g. of H_2

or, 1 e.m.u. of electricity sets free .00010384 g. of H_2.

Now suppose that there are N unit charges in 1 e.m.u. of quantity.

$$1 \text{ e.m.u.} = 3 \times 10^{10} \text{ e.s. unit}$$
$$\therefore Ne = 3 \times 10^{10} \text{ e.s. unit}$$
$$i.e., \ N \times 4.65 \times 10^{-10} = 3 \times 10^{10}$$
$$N = 6.5 \times 10^{19}$$

To get any information as to the mass of the hydrogen atom from these figures we must make an assumption as to the number of unit charges carried by each atom. From other evidence we are led to the belief that just one unit of electricity is associated with each atom of hydrogen.

$$\therefore N = 6.5 \times 10^{19}$$
$$= \text{number of atoms of hydrogen set free by 1 e.m.u.}$$
$$\therefore 6.5 \times 10^{19} \text{ atoms of hydrogen weigh .00010384 gram.}$$
$$\therefore H_a = 1.6 \times 10^{-24} \text{ gram.}$$

22. Number of Molecules of any Gas per Unit Volume.

From the previous section it is possible to determine the weight of an atom of any gas. It is now required to find the number of molecules of any gas in 1 cc. at standard temperature and pressure.

We have the mass of $H_a = 1.65 \times 10^{-24}$ gram.

Also 1 cc. of hydrogen at 0°C. and 76 cms. pressure weighs .00009004 gram.

$$\therefore \text{ number of atoms of hydrogen per 1 cc.} = \frac{.00009004}{1.65 \times 10^{-24}}$$
$$= 5.44 \times 10^{19}$$

or, number of molecules of hydrogen per 1 cc. $= 2.72 \times 10^{19}$
= number of molecules of any gas per 1 cc. $= n$

23. Number of Molecules of Any Gas per Gram-Molecule.

A gram-molecule of any gas is defined to be the molecular weight in grams.

e.g., 1 gram-molecule of H_2 weighs 2 grams.

Now 1 atom of hydrogen weighs 1.65×10^{-24} gram

$\therefore$ 1 molecule of hydrogen weighs 3.3×10^{-24} gram

$\therefore$ number of molecules of hydrogen per gram-molecule

$$= \frac{2.000}{3.3 \times 10^{-24}}$$
$$= 6.2 \times 10^{23}$$

= number of molecules of any gas per gram-molecule
$= N$.

24. Determination of $\frac{e}{m}$ for Electrons.

There are three methods of determining with accuracy the value of $\frac{e}{m}$ for electrons.

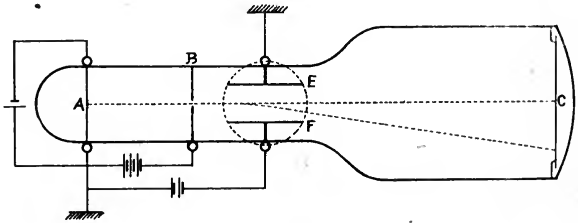

Fig. 12

Method 1.—The experimental arrangement is shown in Fig. 12. A is a strip of platinum coated with calcium oxide, and is called a Wehnelt cathode. When heated, this cathode emits a copious stream of electrons. B is an aluminium disc with a small hole in the centre. The tube containing A and B is exhausted to a very low pressure and sealed off. A and B are connected respectively to the negative and positive poles of a battery, induction coil, or electrical machine. The electrons are accelerated by this field

and some of them pass through the hole in B and strike the flourescent screen C. This stream of electrons passes between two parallel plates E and F, and between the poles of an electro-magnet, the cross-section of which is represented by the dotted circle in the figure.

Let v = velocity of an electron at B,

e = charge on an electron,

m = mass of an electron,

l = length of the plates E and D,

= diameter of the pole-pieces of the magnet,

d = distance from E to C.

If a field H is created by the magnets the electrons will be deflected in passing through this field and will strike the screen at a point D cms. away from the point at which they struck it before. By Laplace's law, the force on unit length of a current flowing perpendicularly to a magnetic field is Hi.

In this case $i = ev$,

$\therefore F = Hev = mf$, where f is the acceleration given to the electrons by the field.

$$\therefore f = \frac{Hev}{m}$$

The electrons will remain in the field for a time $\frac{l}{v}$, and during this time, on account of the force exerted on them by the field, they will move a distance s_1 perpendicular to the field and to the original direction of motion.

$$\therefore s_1 = \tfrac{1}{2} f t^2 = \tfrac{1}{2} \cdot \frac{Hev}{m} \cdot \frac{l^2}{v^2}$$

When the electrons leave the field their acceleration ceases and they will then possess a velocity v_1 in the direction of s_1, given by

$$v_1 = ft = \frac{Hev}{m} \cdot \frac{l}{v}$$

It will take the electrons $\frac{d}{v}$ secs. to traverse the distance d. During this time, on account of the velocity v_1, they will traverse a distance s_2 parallel to s_1 and given by

$$s_2 = \frac{Hev}{m} \cdot \frac{l}{v} \cdot \frac{d}{v}$$

Then the total deflection $D_1 = s_1 + s_2$

$$= \tfrac{1}{2} \cdot \frac{Hev}{m} \cdot \frac{l^2}{v^2} + \frac{Hev}{m} \cdot \frac{l}{v} \cdot \frac{d}{v}$$

$$= \frac{He}{m} \cdot \frac{l}{v} \left(\frac{l}{2} + d \right)$$

Similarly, if instead of the magnetic field, an electric field X is maintained between the plates E and F, the point at which the electrons strike C will be deflected through a certain distance, say D_2 cms.

The force exerted on an electron by the field is Xe,

$$\therefore\ F = Xe = mf, \text{ or } f = \frac{Xe}{m}$$

Proceeding as in the case of the magnetic field we get

$$D_2 = \tfrac{1}{2}\,\frac{Xe}{m}\cdot\frac{l^2}{v^2} + \frac{Xe}{m}\cdot\frac{l}{v}\cdot\frac{d}{v}$$

$$= \frac{Xe}{m}\cdot\frac{l}{v^2}\left(\frac{l}{2} + d\right)$$

Now if the electric and magnetic fields are applied simultaneously and their strengths adjusted until the spot at which the electrons strike C is brought back to its original position we will have

$$D_1 + D_2 = 0,$$

$$i.e.,\ D_1 = -D_2.$$

Equate the absolute values of these deflections, then

$$\frac{He}{m}\cdot\frac{l}{v}\left(\frac{l}{2} + d\right) = \frac{Xe}{m}\cdot\frac{l}{v^2}\left(\frac{l}{2} + d\right)$$

$$\therefore\ v = \frac{X}{H}$$

Substitute in the value of D_1 found above.

$$D_1 = \frac{He}{m}\cdot\frac{lH}{X}\left(\frac{l}{2} + d\right)$$

$$\therefore\ \frac{e}{m} = \frac{D_1 X}{H^2 l\left(\frac{l}{2} + d\right)}$$

Method 2.—The apparatus is shown in figure 13. The electrons used here are cathode rays which are produced by an electrical machine *EM* connected between *A* and *B*. The electrons pass through a small hole at *B* into a metal cylinder *C* and impinge on the screen *D*. The metal cylinder protects the electrons from any external electrostatic influences. *C* is placed between the poles of a large magnet *SS*, so that a magnetic field *H* may be applied to the electrons. The apparatus can be exhausted to a suitable pressure through the tube *T*. When the magnetic field is applied the point at which the electrons impinge on the screen *D* will be shifted, and from this we may calculate the ratio of the charge to the mass of the electrons in the stream.

Let e, m, v, have the same significance as before

D = distance through which point of impact is moved

l = length of path of the electrons in the magnetic field

V_o = difference of potential between A and B.

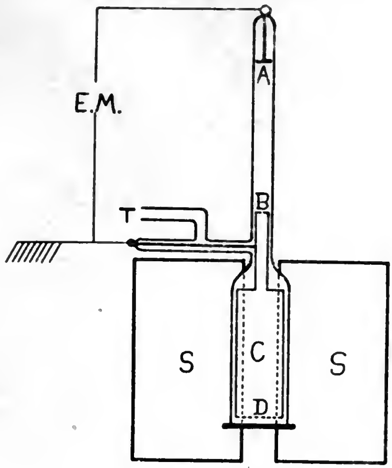

Fig. 13

The work done in accelerating an electron as it travels from A to B will be eV_o, and this work must be equal to the kinetic energy of the electron at B.

Then $\frac{1}{2} mv^2 = eV_o$

$$v = \sqrt{\frac{2eV_o}{m}}$$

The time during which the electron is moving in the magnetic field is $t = \frac{l}{v} = \frac{l}{\sqrt{\frac{2eV_o}{m}}}$

As before the acceleration given to the electron by the magnetic field is $f = \frac{Hev}{m}$

Therefore $D = \frac{1}{2} ft^2$

$$= \frac{1}{2} \frac{He}{m} \cdot \sqrt{\frac{2eV_o}{m}} \cdot \frac{l^2}{\frac{2eV_o}{m}}$$

$$= \frac{1}{2} Hl^2 \cdot \sqrt{\frac{e}{2mV_o}}$$

Therefore $$\frac{e}{2mV_o} = \left(\frac{2D}{Hl^2}\right)^2$$

and $$\frac{e}{m} = \frac{8D^2}{l^4} \cdot \frac{V_o}{H^2}$$

Method 3.—The apparatus is shown in figure 14. Two small bell-jars are placed on opposite sides of a plate BC. The upper one of these contains a Wehnelt cathode A, which is placed directly behind a small hole in BC. DE is a photographic plate fastened to BC, and having a small hole coinciding with that in BC. This plate is covered by a metal cap F which protects the electrons in the lower bell-jar from any external electrostatic effects. The

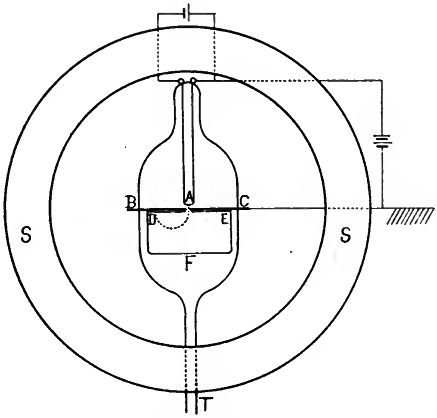

Fig. 14

whole apparatus is placed at the centre of a large solenoid, represented in section at SS, by means of which a magnetic field may be applied to the electrons which leave A. The bell-jars can be exhausted to a suitable pressure through the tube T.

A difference of potential is maintained between BC and A. The electrons will be accelerated by this field, pass through the hole in BC, be curled up by the magnetic field H, and will make traces on the photographic plate wherever they impinge on it.

Then, as in method 2, $eV_o = \frac{1}{2}mv^2$

$$v^2 = \frac{2eV_o}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The magnetic field will exert a force Hev on an electron, and under this force the electron will move in a circle of radius ρ, where ρ is given by $Hev = \frac{mv^2}{\rho}$

From the above $\frac{He\rho}{m} = v$

and $\frac{H^2e^2\rho^2}{m^2} = v^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Then from (1) and (2) $2\frac{e}{m}V_\circ = \frac{H^2\rho^2e^2}{m^2}$

$$\frac{e}{m} = \frac{2V_\circ}{H^2\rho^2}$$

ρ may be found from the trace of the electrons on the photographic plate.

Using these three methods many determinations of the ratio $\frac{e}{m}$ for electrons have been made. The mean value obtained for this ratio is 1.77×10^7, when e is expressed in electromagnetic units, or, 5.31×10^{17}, when e is expressed in electrostatic units.

25. Mass of the Electron.

From the above $\frac{e}{m}=5.31\times10^{17}$ E.S.U.

But $e=4.65\times10^{-10}$ E.S.U.

Therefore $m = \frac{4.65\times10^{-10}}{5.31\times10^{17}}$ grams

$=8.8\times10^{-28}$ grams.

That is, mass of the electron $= \frac{1}{1800}$ mass of the Hydrogen atom.

26. Electrical Units.

There are three systems of electrical units, the electrostatic, the electromagnetic, and the practical systems. Each of the systems is complete in itself and has its own particular uses. The electrostatic system is based on the mutual force exerted by electric charges; the electromagnetic system on the mutual force exerted by magnetic poles; and the practical system on the chemical action of an electric current.

I. *The Electrostatic System.*

The electrostatic magnitudes will be denoted by capital letters; the capital letter enclosed in square brackets denoting the unit of the particular magnitude under consideration.

(*a*) $[Q]$ = unit of quantity.

$[Q]$ is determined by Coulomb's law

$$F = \frac{e_1\,e_2}{r^2}$$

Then $[Q]$ is defined to be that quantity of electricity which, when placed in a vacuum, at a distance of 1 cm. from an equal and similar quantity is repelled with a force of 1 dyne.

(*b*) $[I]$ = unit of current

$$= \frac{[Q]}{t}$$

(*c*) $[M]$ = unit magnetic pole.

If $[I]$ be passed through a wire 1 cm. long bent into the form of an arc of a circle of 1 cm. radius, then the unit pole is that which when placed at the centre of the circle is acted upon by a force of 1 dyne.

(*d*) $[E]$ = unit of potential difference.

Two charged bodies are at unit potential difference when it requires one erg of work to take $[Q]$ from the one at lower potential to the one at higher potential.

(*e*) $[R]$ = unit of resistance.

= resistance of a circuit in which $[E]$ will maintain a current $[I]$.

(*f*) $[C]$ = unit of capacity

= quantity of electricity required to raise a body to unit potential, $\therefore [C] = \frac{[Q]}{[E]}$

II. *The Electromagnetic System.*

The electromagnetic magnitudes will be denoted by small letters; the small letter enclosed in square brackets denoting the unit of the particular magnitude under consideration.

(*a*) $[m]$ = Unit magnetic pole. The unit magnetic pole is determined by the law $F = \frac{m_1 m_2}{r^2}$ which is analogous to that of Coulomb.

Then the unit pole is defined to be that pole which, when placed at a distance of 1 cm. from an equal and similar pole, repels it with a force of 1 dyne.

(*b*) $[i]$ = unit of current.

$[i]$ is defined to be that current which, when passed through a wire 1 cm. long bent into the arc of a circle of 1 cm. radius, exerts a force of 1 dyne on a unit pole at its centre.

(*c*) $[q]$ = unit of quantity

= $[i]\, t$.

(*d*) $[e]$ = unit of potential difference.

(*e*) $[r]$ = unit of resistance.

(*f*) $[c]$ = unit of capacity.

The definitions of the last three units are entirely analogous to those of the corresponding units in the e.s. system.

III. *Comparison of the Electrostatic and Electromagnetic Systems.*

It is evident from the above definitions that the units in the two systems are not at all of the same size. In 1856 Weber and Kohlrausch measured the ratio of $[q]$ to $[Q]$; they denoted this ratio by v. Thus we have as a fundamental equation for comparing the two systems

(*a*) $$[q] = v\,[Q].$$

The ratios of all the other corresponding units are found to be simple functions of v.

(*b*) From (*a*) it at once follows that

$$[i] = v\,[I].$$

(*c*) A little consideration will show that the ratio of the unit poles in the two systems must be the inverse of the ratio of the unit currents,

$$i.e., [m] = \frac{1}{v}\,[M].$$

(*d*) From the definition of $[E]$ it follows that

$$[Q]\,[E] = 1 \text{ erg}.$$

Similarly, $$[q]\,[e] = 1 \text{ erg}.$$

$$\therefore [q]\,[e] = [Q]\,[E]$$

and, $$\frac{[e]}{[E]} = \frac{[Q]}{[q]} = \frac{1}{v}$$

or, $$[e] = \frac{1}{v}[E].$$

(*e*) From Ohm's law we have

$$[R] = \frac{[E]}{[I]}, \text{ and } [r] = \frac{[e]}{[i]}$$

$$\therefore \frac{[r]}{[R]} = \frac{[e]}{[i]} \cdot \frac{[I]}{[E]} = \frac{[e]}{[E]} \cdot \frac{[I]}{[i]} = \frac{1}{v^2}.$$

(*f*) From the definition of the units of capacity we have

$$[C] = \frac{[Q]}{[E]}, \text{ and } [c] = \frac{[q]}{[e]}$$

$$\therefore [c] = v^2[C].$$

IV. *The Dimensions of v.*

The question arises as to whether v is just a number, or whether it possess physical significance. To decide this question we must find the dimensions of v, that is, the way in which the units of force, length and time enter into it. We will use the letters F, L, T to denote dimensions in force, length and time.

We have seen that $[q] = v[Q]$; hence if we have a definite quantity of electricity which contains q/e.m. units or Q/e.s. units

it is evident that $q = \frac{1}{v} Q$. Also Coulomb's law states that $\frac{Q^2}{r^2} = F$.
Therefore Q has the dimensions $LF^{1/2}$.

Again, from the theory of the tangent galvanometer, we have

$$i = \frac{Ha \tan \theta}{2\pi n}, \text{ (see section 31)}$$

where i is measured electromagnetically; therefore i has the dimensions HL. Now consider a magnetic pole of strength m, lying in a magnetic field H; the force on the pole is Hm. Then H has the dimensions $\frac{F}{m}$

But from the magnetic analogue of Coulomb's law

$$\frac{m^2}{r^2} = F,$$

$\therefore$ m has the dimensions $F^{1/2}L$.
Combining these results,

i has the dimensions $F^{1/2}$,

and q has the dimensions $F^{1/2}T$.

$$\text{Then } v = \frac{Q}{q} = \frac{LF^{1/2}}{F^{1/2}T} = \frac{L}{T},$$

i.e., v has the dimensions of a velocity. It has been shown that v has the same numerical value as the velocity of light, and the physical significance of this fact has been demonstrated in the electromagnetic theory of light as developed by Maxwell and J. J. Thomson.

The value of v may be determined experimentally by comparative measurements of a quantity of electricity, of a capacity, or of a potential difference, in the two systems. Also, since the identity of v with the velocity of light has been established, v may be evaluated by measuring that velocity. A list of the most accurate determinations of v is appended.

Experimenter	*Date*	*Result*
Himstedt	1886, 1887, 1888	3.0057×10^{10}
Rosa	1889	3.0000×10^{10}
Thomson	1890	2.9960×10^{10}
Pellat	1891	3.0010×10^{10}
Abraham	1892	2.9913×10^{10}
Hermuzesen	1896	2.9973×10^{10}
Fabry and Perot	1898	3.0001×10^{10}

V. *The Practical System.*

Several of the units in the e.m. and e.s. systems are inconveniently large for practical work, several are too small. So at a congress at Paris in 1881 a practical system employing units of convenient size was agreed upon. These units were so chosen that they could be easily expressed in terms of the units of the two systems previously discussed. For practical purposes however, they are defined as follows:

(*a*) the coulomb = unit of quantity. This is the quantity of electricity which will deposit 0.001118 gram of silver from a solution of a silver salt.

(*b*) the ampere = unit of current. A circuit is said to be conducting unit current when 1 coulomb traverses a cross-section of the circuit in 1 second.

(*c*) the ohm = unit of resistance. The ohm is defined as the resistance at 0° C. of a column of mercury, 1 sq. mm. in cross-section and 106.3 cms. long.

(*d*) the volt = unit of potential difference. If from any circuit carrying a current of 1 ampere, we select a portion having a resistance of 1 ohm, the potential difference between the ends of this portion will be 1 volt.

(*e*) the farad = unit of capacity. The farad is the capacity of a condenser which contains 1 coulomb of electricity when the potential difference between its terminals is 1 volt. The farad is too large a unit for practical work, so the unit usually used is the micro-farad which is one-millionth of a farad.

The following table expresses the units of the e.s. and practical systems in terms of those of the e.m. system.

Magnitude	e.s. Unit	Practical System	
		Unit	Relation
Quantity	$\frac{1}{v}[q]$	coulomb	$\frac{1}{10}[q]$
Current	$\frac{1}{v}[i]$	ampere	$\frac{1}{10}[i]$
Potential diff.	$v[e]$	volt	$10^8[e]$
Resistance	$v^2[r]$	ohm	$10^9[r]$
Capacity	$\frac{1}{v^2}[c]$	farad	$10^{-9}[c]$

MAGNETISM.

27. Determination of the Horizontal Intensity of the Earth's Magnetism.

This determination involves two separate series of observations, the first being the measurement of the period of swing of a suspended magnet; and the second, the measurement of the deflection produced by this magnet on a small second magnet suspended so as to lie along a line drawn at right angles to the middle point of the first.

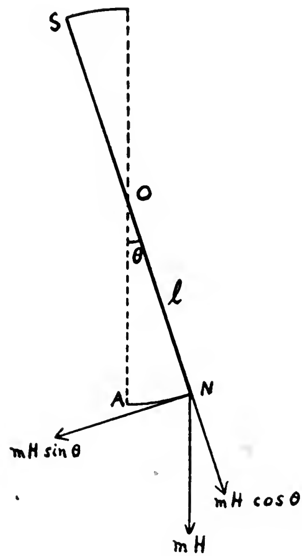

Fig. 15

(1) Let NS (Fig. 15) be a suspended magnet of length $2l$ and pole-strength m, and let it be given a small angular displacement θ.

The force acting at $N = mH$,

where $H =$ horizontal intensity of the earth's magnetism.

The part of this force which tends to cause rotation is $mH \sin \theta$, and

for small displacements $\sin \theta = \theta = \dfrac{AN}{l}$

$\therefore$ effective force on $N = mH \cdot \dfrac{AN}{l}$

i.e., the restoring force is proportional to the displacement, and therefore the magnet oscillates like a simple pendulum with simple harmonic motion.

The period of oscillation of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{\omega g l}}$$

where I = moment of inertia,

ω = mass.

g = constant of gravitation.

In the case of a magnet, ωg is replaced by an attraction $+mH$ and a repulsion $-mH$, or by a total force $2mH$.

$$\therefore T = 2\pi\sqrt{\frac{I}{2mHl}}$$

$$= 2\pi\sqrt{\frac{I}{MH}}$$

where $2ml = M$

= moment of the magnet.

$$\therefore MH = \frac{4\pi^2 I}{T^2}.$$

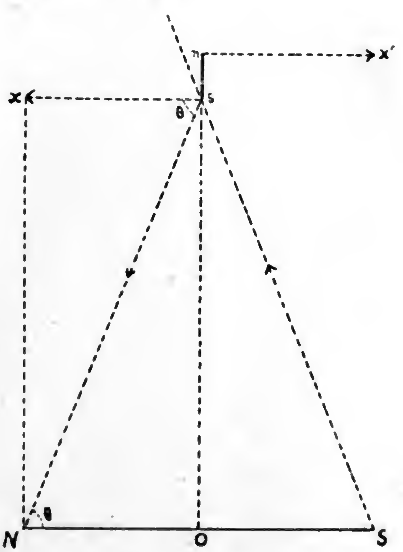

Fig. 16

(2) Let NS (Fig. 16) be the magnet previously used, and ns the small magnet, and let the pole-strengths be respectively m and m'.

The force of attraction on s due to $N = \dfrac{mm'}{(Ns)^2}$

$$= \frac{mm'}{d^2 + l^2}$$

where d = distance sO = distance nO
if ns is small compared with sO.

The turning force acting on s is then $\dfrac{mm'}{d^2+l^2}\cos\theta$.

Similarly, the effective force of repulsion on s due to S

$$= \frac{mm'}{d^2+l^2}\cos\theta.$$

Or, total effective turning force on s, along $sx = \dfrac{2mm'}{d^2+l^2}\cos\theta$

and total effective turning force on n, along $sx' = \dfrac{2mm'}{d^2+l^2}\cos\theta$.

The moment of the couple acting on $ns = \dfrac{2mm'}{d^2+l^2}\cos\theta \,.\, 2\lambda$,

where 2λ = length ns

$$\therefore \text{ moment} = \frac{2mm'}{d^2+l^2} \cdot \frac{l}{(d^2+l^2)^{1/2}} \,.\, 2\lambda$$

$$= \frac{Mm'}{(d^2+l^2)^{3/2}} \,.\, 2\lambda.$$

The magnet ns will be deflected until the moment of the resultant couple is equal to the moment of the couple due to the field H. Hence, if the deflection $= \alpha$,

$$\text{moment due to } H = m'H \,.\, 2\lambda \,.\, \sin\alpha,$$

and resultant moment due to $NS = \dfrac{Mm'}{(d^2+l^2)^{3/2}} \,.\, 2\lambda \,.\, \cos\alpha$.

$$\therefore \frac{Mm'}{(d^2+l^2)^{3/2}} \,.\, 2\lambda \,.\, \cos\alpha = m'H \,.\, 2\lambda \,.\, \sin\alpha$$

$$\therefore \frac{M}{H} = (d^2+l^2)^{3/2}\tan\alpha$$

From these two observations we have

$$(1)\quad MH = \frac{4\pi^2 I}{T^2}$$

$$(2)\quad \frac{M}{H} = (d^2+l^2)^{3/2}\tan\alpha.$$

Dividing (1) by (2), $H^2 = \dfrac{4\pi^2 I}{T^2(d^2+l^2)^{3/2}\tan\alpha}$.

28. Laplace's Law for the Action of an Element of an Electric Circuit on a Magnetic Pole.

It is never possible to obtain an element of current without having a complete circuit, nevertheless, for many calculations it is very convenient to have an expression representing the theoretical effect of an element.

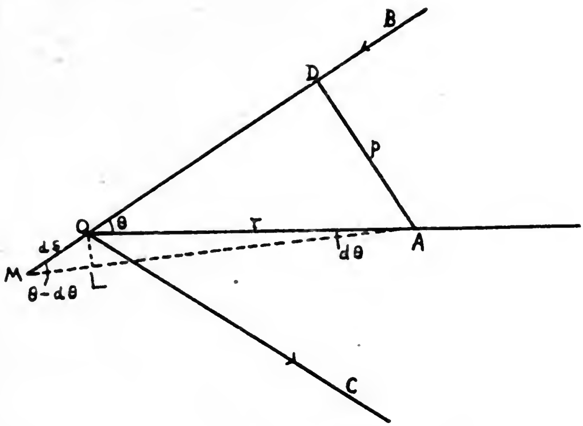

Fig. 17

Consider a linear circuit of the form BOC (Fig. 17) which makes the angle 2θ at O, and draw OA in the plane of BOC so as to bisect the angle BOC. From A draw AD perpendicular to OB and of length p, and let the length of OA be r.

According to Biot and Savart's law the action of this circuit on a unit pole at A is given by

$$F = \frac{k \tan \frac{\theta}{2}}{r},$$

where k is some constant.

$$i.e., F = \frac{k \tan \frac{\theta}{2} \sin \theta}{p}$$

$$= \frac{2k \sin^2 \frac{\theta}{2}}{p}$$

and by the symmetry of the circuit this will act along OA.

The magnetic intensity due to one branch alone must be

$$F_1 = \tfrac{1}{2} F$$

$$= \frac{k \sin^2 \frac{\theta}{2}}{p}$$

Let now the branch BO be increased by a small element OM of length ds. Join MA and draw OL perpendicular to MA. The intensity at A will now be increased by a small amount dF_1,

$$\text{such that } dF_1 = \frac{2k \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cdot \tfrac{1}{2} d\theta}{p}$$

$$= \frac{k \cdot 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta}{2p}$$

$$= \frac{k \sin \theta d\theta}{2p}$$

Now $p = r \sin \theta$.

$OL = rd\theta$ (nearly);

also, $OL = OM \sin (\theta - d\theta)$

$= ds \sin \theta$.

$$\therefore d\theta = \frac{ds \sin \theta}{r}$$

$$\therefore dF_1 = \frac{kds \sin^2 \theta}{2r^2 \sin \theta}$$

$$= \frac{kds \sin \theta}{2r^2}$$

This is the effect of unit current on unit magnetic pole, so that for a current i and pole-strength μ,

$$dF = \frac{k\mu ids \sin \theta}{2r^2}$$

To evaluate k, take the case of a unit current in a circular circuit of 1 cm. radius. Then 1 cm. of the circuit exerts unit force on a unit pole at the centre.

$$\begin{aligned} \text{Then } dF &= 1, \\ \mu &= 1, \\ i &= 1, \\ ds &= 1, \\ \theta &= 90^\circ, \\ \sin \theta &= 1, \\ r &= 1, \\ \therefore \frac{k}{2} &= 1, \\ \text{or, } k &= 2, \end{aligned}$$

$\therefore$ for the action of an element of a circuit we have

$$dF = \frac{\mu ids \sin \theta}{r^2}$$

29. Magnetic Field Due to a Current in an Infinitely Long Wire.

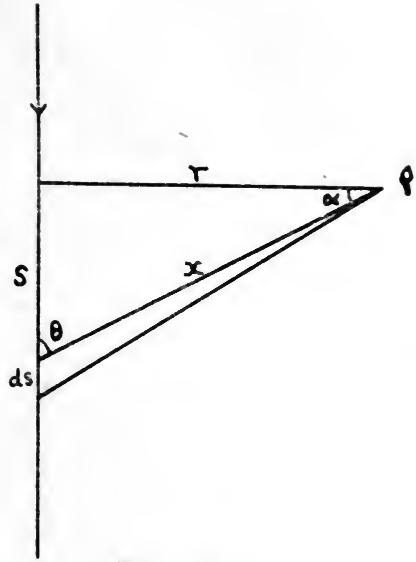

Fig. 18

Let s (Fig. 18) represent a portion of a wire carrying a current, and let an element ds make an angle α with a normal to the wire at a point distant r along the normal.

$$s = r \tan \alpha$$

$$ds = r \sec^2 \alpha d\alpha.$$

By Laplace's equation $dF = \dfrac{\mu i ds \sin \theta}{r^2}$,

and here, $\sin \theta = \cos \alpha$,

$$x = r \sec \alpha,$$

∴ Laplace's equation becomes

$$dF = \frac{\mu i r \sec^2 \alpha d\alpha \cos \alpha}{r^2 \sec^2 \alpha}$$

$$= \frac{\mu i \cos \alpha d\alpha}{r}$$

$$\therefore F = 2 \int_0^{\frac{\pi}{2}} \frac{\mu i \cos \alpha d\alpha}{r}$$

$$= \frac{2 \mu i}{r}$$

or, magnetic intensity $= H = \dfrac{2i}{r}$.

30. Magnetic Field due to a Circular Current.

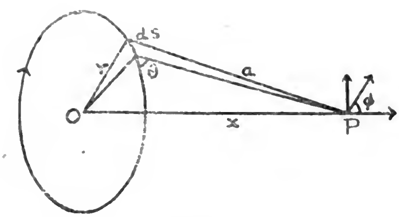

Fig. 19

Every element ds of the circular current (Fig. 19) exerts a force in a different direction at P, so that all the forces due to the complete circuit form a cone with apex at the point P. Each of these will have a component at right angles to the axis, and one along the axis, the former components cancelling each other in pairs. By Laplace's law the force along the axis exerted by each element of the circuit on a pole of strength μ placed at P is

$$dF = \frac{\mu i ds \sin\theta}{a^2} \cdot \cos\phi$$

$$= \frac{\mu i ds \cos\phi}{a^2}, \text{ since } \theta = \frac{\pi}{2}$$

$$= \frac{\mu i ds \cdot r}{a^3}$$

Therefore the force due to whole circuit is

$$F = \frac{\mu i r}{a^3} \cdot 2\pi r$$

$$= \frac{2\pi\mu i r^2}{(r^2+x^2)^{\frac{3}{2}}}$$

Therefore at P, $H = \dfrac{2\pi i r^2}{(r^2+x^2)^{\frac{3}{2}}}$

At O, $x=0$, therefore $H = \dfrac{2\pi i}{r}$

or, for n turns in the circuit $H = \dfrac{2\pi n i}{r}$

31. The Tangent Galvanometer.

The tangent galvanometer consists of a large circular coil of wire with its plane in the direction of the earth's magnetic field, and having a short magnetic needle suspended at its centre. If H is the value of the horizontal component of the earth's field, and if the needle is deflected through an angle θ when a current i is

passing through the coil, the effective component of the restoring force is $H \sin\theta$, while that of the deflecting force is $F\cos\theta$

Therefore $F\cos\theta = H\sin\theta$

$$\text{or } F = H\tan\theta$$

$$\text{Therefore } \frac{2\pi n i}{a} = H\tan\theta$$

$$\text{or } i = \frac{Ha\tan\theta}{2\pi n}$$

This is often written $i = \dfrac{H}{G}\tan\theta$

where C is called the true constant of the galvanometer.

32. Magnetic Field due to a Solenoid.

(1) At a point outside the solenoid (Fig. 20).

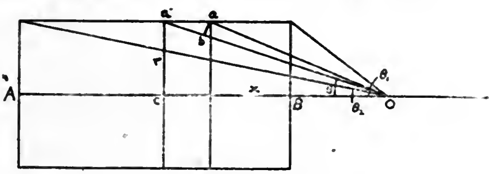

Fig. 20

If, i is the current in the solenoid and there are n_1 turns of wire per unit length, then the amount of current in the solenoid per unit length is $i' = n_1 i$, and the total current in a small section aa' is $i'aa'$. Then the force on unit pole at O is, by the development of Laplace's law in section 30

$$dF = \frac{2\pi r^2 i' aa'}{(r^2+x^2)^{\frac{3}{2}}}$$

$$\text{Now } \frac{aa'}{ab} = \frac{a'O}{r}$$

and $ab = a'O\,d\theta$.

$$\text{Therefore } aa' = \frac{\overline{a'O}^2 \,.\, d\theta}{r} = \frac{r^2+x^2}{r}\,d\theta$$

$$\text{Therefore } dF = \frac{2\pi r^2 i'}{(r^2+x^2)^{\frac{3}{2}}} \cdot \frac{r^2+x^2}{r} \,.\, d\theta$$

$$= \frac{2\pi r\, i'\, d\theta}{(r^2+x^2)^{\frac{1}{2}}}$$

$$= 2\pi i' \sin\theta\, d\theta.$$

$$\text{Therefore } F = \int_{\theta_2}^{\theta_1} 2\pi i' \sin\theta\, d\theta.$$

$$= 2\pi i'(\cos\theta_1 - \cos\theta_2).$$

(2) At a point at the centre of the solenoid (Fig. 21).

$$\text{Here } \theta_1 = \pi - \theta_2$$

Therefore $F = 2\pi i'(\cos\theta_1 + \cos\theta_1)$

$$= 4\pi i' \cos\theta_1$$

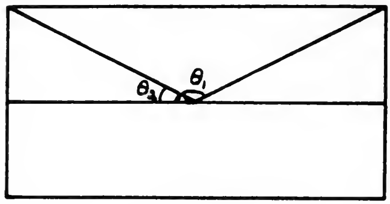

Fig. 21

If there are n turns of wire in the whole solenoid and the length of the solenoid is l.

$$\text{then } i' = \frac{ni}{l}$$

Therefore $F = \dfrac{4\pi ni \cos\theta}{l}$

For a long solenoid $\theta = \pi$

Therefore $F = \dfrac{4\pi ni}{l}$ at any point which is distant from the ends of the solenoid.